Amendment on last Tutorial

(See Wiki Page "Spectral Radius" For more details)

"$\|\vec{e}_k\| \leq \rho(M)^k \|\vec{e}_0\|$"

Not True in general.

But true as $k \to \infty$.

$$\|\vec{e}_k\| = \|M^k \vec{e}_0\|$$

$$\leq \|M^k\| \, \|\vec{e}_0\| \qquad \left(\|A\| \overset{\text{def}}{=\!=} \sup_{\vec{x} \neq 0} \frac{\|A\vec{x}\|}{\|\vec{x}\|} \right)$$

Gelfand's Formula: $\rho(M) = \lim_{k \to \infty} \|M^k\|^{1/k}$

So, $\|\vec{e}_k\| \leq \|M^k\| \, \|\vec{e}_0\|$

$$\to \rho(M)^k \|\vec{e}_0\| \text{ as } k \to \infty$$

Explanation on HW3 Q(5)(b).

5. Consider the following iterative scheme:

$$x_{k+1} = (\alpha I - tA)x_k + tb$$

where $\alpha \geq 1$. Suppose that A is symmetric positive definite matrix in $\mathbb{R}^{n\times n}$, with eigenvalues $\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 > 0$.

(a) Show that the above scheme converges if and only if $\frac{\alpha-1}{\lambda_1} < t < \frac{\alpha+1}{\lambda_n}$.

(b) Prove that the optimal t, in the sense of rate of convergence, is $\frac{2\alpha}{\lambda_1+\lambda_n}$

Matrix $\alpha I - tA$ has eigenvalues $\alpha - t\lambda_j$

If $t < 0$,

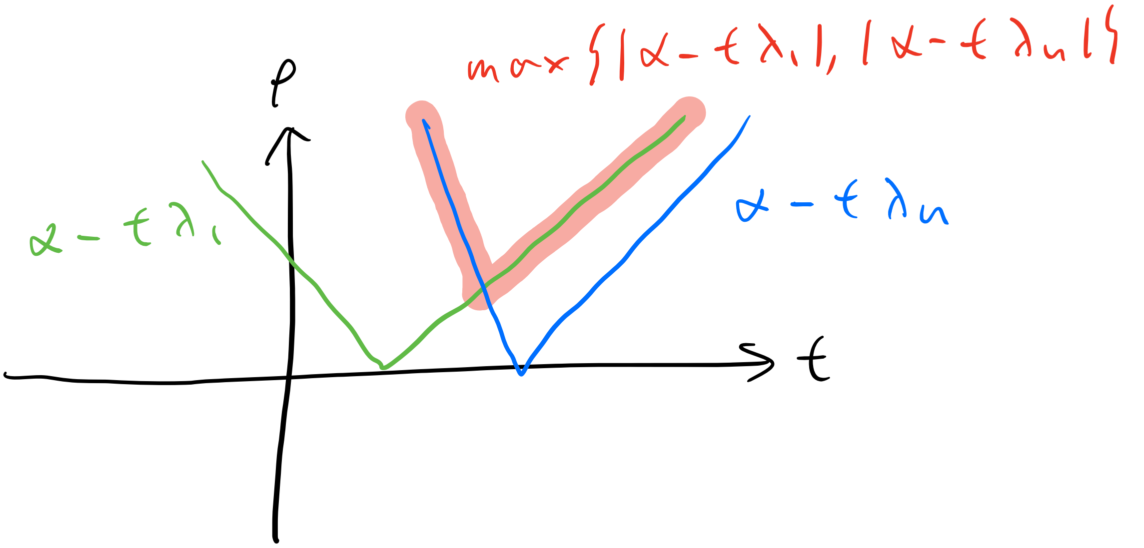
$$\alpha - t\lambda_1 \leq \alpha - t\lambda_j \leq \alpha - t\lambda_n$$

If $t > 0$,

$$\alpha - t\lambda_n \leq \alpha - t\lambda_j \leq \alpha - t\lambda_1$$

In any cases,

$$|\alpha - t\lambda_j| \leq \max\{|\alpha - t\lambda_1|, |\alpha - t\lambda_n|\}$$
$$= \rho(\alpha I - tA).$$



$\therefore$ optimal t satisfy $|\alpha - t\lambda_1| = |\alpha - t\lambda_n|$.

Power Method

$A \in \mathbb{C}^{n \times n}$

A has eigenvalues $|\lambda_1| > |\lambda_2| \geq \cdots |\lambda_n| \geq 0$

and $\vec{v}_1$ is a eigenvector w.r.t. λ_1.

Initial Guess:

$\vec{x}^0 = a_1 \vec{v}_1 + (\text{ something else })$, $a_1 \neq 0$.

(Random choice of $\vec{x}_0$ is usually OK)

And iterative scheme:

$$\vec{x}^{k+1} = \frac{A \vec{x}^k}{\| A \vec{x}^k \|_\infty}$$

where $\| \vec{v} \|_\infty = \left\| \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|_\infty = \max\limits_{1 \leq i \leq n} \{ |v_i| \}$

Then $\| A \vec{x}^k \|_\infty \to |\lambda_1| = \rho(A)$

as $k \to \infty$

(See Lecture Notes).

Remark

$\|\cdot\|_\infty$ is not important,

can be replaced by other norm.

e.g. $$\vec{x}^{k+1} = \frac{A\vec{x}^k}{\|A\vec{x}^k\|_2},$$

and $\|A\vec{x}^k\|_2$ also converges

to $|\lambda_1| = \rho(A)$.

pf exactly the same steps in

lecture notes

Geometric Understanding (Past Homework).

$A \in \mathbb{C}^{n\times n}$ is Normal $(AA^* = A^*A)$

(i.e. A is diagonalizable.)

with eigenvalues:

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0.$$

and eigenvectors: $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$.

$$A = QDQ^*, \quad Q^*Q = I$$

$$Q = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

<u>Def</u>: For $\vec{x}, \vec{y} \in \mathbb{C}^n$,

$$\cos\angle(\vec{x},\vec{y}) = \frac{|\langle \vec{x},\vec{y}\rangle|}{\|\vec{x}\|_2 \|\vec{y}\|_2}$$

$$\sin\angle(\vec{x},\vec{y}) = \sqrt{1-\cos^2\angle(\vec{x},\vec{y})}$$

$$\tan\angle(\vec{x},\vec{y}) = \frac{\sin\angle(\vec{x},\vec{y})}{\cos\angle(\vec{x},\vec{y})}.$$

Recall $\langle \vec{x},\vec{y}\rangle = \sum x_i \overline{y_i}$

Consider the iterative scheme

$$\vec{x}^{k+1} = A\vec{x}^k = A^k \vec{x}^0$$

F_1: $\cos\angle(\mathcal{Q}^*\vec{x}, \mathcal{Q}^*\vec{y}) = \cos\angle(\vec{x}, \vec{y})$.

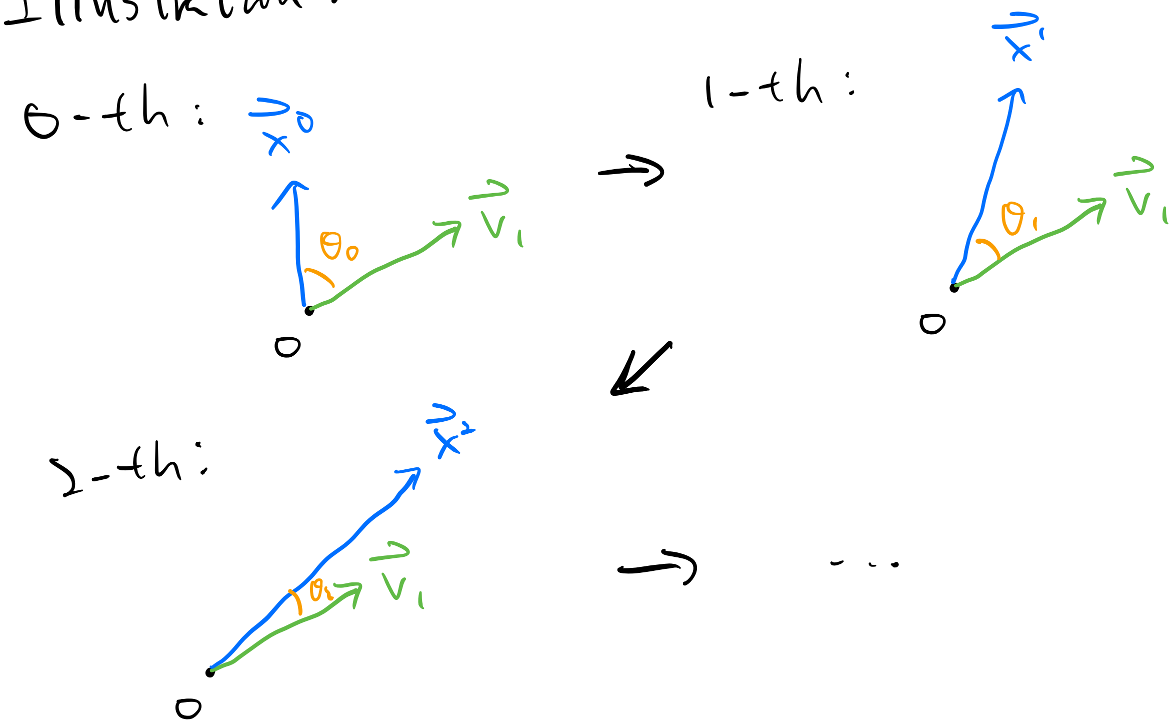
$$
\begin{aligned}
&\langle \mathcal{Q}^*\vec{x}, \mathcal{Q}^*\vec{y} \rangle \\
&= (\mathcal{Q}^*\vec{y})^* (\mathcal{Q}^*\vec{x}) \\
&= \vec{y}^* \mathcal{Q}\mathcal{Q}^* \vec{x} \\
&= \langle \vec{x}, \vec{y} \rangle
\end{aligned}
\qquad
\begin{aligned}
&\|\mathcal{Q}^*\vec{x}\|_2^2 \\
&= \langle \mathcal{Q}^*\vec{x}, \mathcal{Q}^*\vec{x} \rangle \\
&= \langle \vec{x}, \vec{x} \rangle \\
&= \|\vec{x}\|_2^2
\end{aligned}
$$

F_2: If $\cos\angle(\vec{x}^0, \vec{v}_1) \neq 0$,

then

$$\tan\angle(\vec{x}^{k+1}, \vec{v}_1) \leq \frac{|\lambda_2|}{|\lambda_1|} \tan\angle(\vec{x}^k, \vec{v}_1).$$

Illustration:



pf (F_2):

$$\vec{x}^{k+1} = A\vec{x}^k$$

$$\Rightarrow \quad \vec{x}^{k+1} = \mathcal{Q} D \mathcal{Q}^* \vec{x}^k$$

$$\Rightarrow \quad \mathcal{Q}^* \vec{x}^{k+1} = D(\mathcal{Q}^* \vec{x}^k)$$

$$\Rightarrow \quad \hat{x}^{k+1} = D\hat{x}^k$$

$$\cos^2\angle(\vec{x}^k, \vec{v}_1)$$

$$= \cos^2\angle(\mathcal{Q}^*\vec{x}^k, \mathcal{Q}^*\vec{v}_1)$$

$$= \cos^2\angle(\hat{x}^k, \vec{e}_1) \qquad \left(\mathcal{Q}^*\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{e}_1\right)$$

$$= \frac{|\langle \hat{x}^k, \vec{e}_1\rangle|^2}{\|\hat{x}^k\|_2^2 \, \|\vec{e}_1\|_2^2}$$

$$= \frac{|(\hat{x}^k)_1|^2}{\|\hat{x}^k\|_2^2}$$

$$\sin^2\angle(\vec{x}^k, \vec{v}_1)$$

$$= 1 - \cos^2\angle(\vec{x}^k, \vec{v}_1)$$

$$= \frac{\|\hat{x}^k\|_2^2 - |(\hat{x}^k)_1|^2}{\|\hat{x}^k\|_2^2}$$

$$= \frac{\sum_{i=2}^{n} |(\hat{x}^k)_i|^2}{\|\hat{x}^k\|_2^2}$$

$$\tan^2 \angle(\vec{x}^k, \vec{v}_1)$$

$$= \frac{\sin^2 \angle(\vec{x}^k, \vec{v}_1)}{\cos^2 \angle(\vec{x}^k, \vec{v}_1)}$$

$$= \frac{\sum_{i=1}^{n} |(\hat{x}^k)_i|^2}{|(\hat{x}^k)_1|^2}$$

$$= \frac{\sum_{i=1}^{n} |(D\hat{x}^{k-1})_i|^2}{|(D\hat{x}^{k-1})_1|^2}$$

$$= \frac{\sum_{i=1}^{n} |\lambda_i (\hat{x}^{k-1})_i|^2}{|\lambda_1 (\hat{x}^{k-1})_1|^2}$$

$$\leq \frac{|\lambda_2|^2 \sum_{i=1}^{n} |(\hat{x}^{k-1})_i|^2}{|\lambda_1|^2 \, |(\hat{x}^{k-1})_1|^2}$$

$$= \left|\frac{\lambda_2}{\lambda_1}\right|^2 \tan^2 \angle(\vec{x}^{k-1}, \vec{v}_1)$$

□

Exercise (Power Method with Shift).

With the above notation,

let $\mu \in \mathbb{R}$, $A - \mu I$ invertible,

$|\lambda_1 - \mu| < |\lambda_2 - \mu| \leq \cdots \leq |\lambda_n - \mu|$,

consider the iterative scheme:

$$\vec{x}^{k+1} = (A - \mu I)^{-1} \vec{x}^{k}$$

Show

$$\tan \angle (\vec{x}^{k+1}, \vec{v}_1) \leq \frac{|\lambda_1 - \mu|}{|\lambda_2 - \mu|} \tan \angle (\vec{x}^{k}, \vec{v}_1)$$

Solution:

Let $\hat{A} = (A - \mu I)^{-1}$

Note $A - \mu I$

has eigenvalues: $\lambda_1 - \mu, \lambda_2 - \mu, \cdots, \lambda_n - \mu$

$$|\lambda_1 - \mu| < |\lambda_2 - \mu| \leq \cdots \leq |\lambda_n - \mu|$$

and shares the same eigenvectors with A.

$\therefore$ $\hat{A}$ has eigenvalue:

$$\hat{\lambda}_1 = \frac{1}{\lambda_1 - \mu}, \quad \hat{\lambda}_2 = \frac{1}{\lambda_2 - \mu}, \quad \cdots, \quad \hat{\lambda}_n = \frac{1}{\lambda_n - \mu}$$

$$|\hat{\lambda}_1| > |\hat{\lambda}_2| \geq \cdots \geq |\hat{\lambda}_n|$$

$$(A - \mu I)\vec{v}_1 = (\lambda_1 - \mu)\vec{v}_1$$

$$\Rightarrow \quad \frac{1}{\lambda_1 - \mu}\vec{v}_1 = (A - \mu I)^{-1}\vec{v}_1$$

$\therefore$ $\hat{A}$ and A share the same eigenvectors.

So, by (F_2),

$$\tan\angle(\vec{x}^{k+1}, \vec{v}_1) \leq \frac{|\hat{\lambda}_2|}{|\hat{\lambda}_1|}\tan\angle(\vec{x}^k, \vec{v}_1)$$

$$= \frac{|\lambda_1 - \mu|}{|\lambda_2 - \mu|}\tan\angle(\vec{x}^k, \vec{v}_1)$$